# POEMS 2022
# PROGRAM AND ABSTRACTS

Version of December 13, 2022

# Program

**Monday 12 December**

9:45-10:00 **Opening**

10:00-10:25 **Andrea Cangiani** *Virtual element method for nondivergence form elliptic equations.* Chair: Marco Verani

10:25-11:10 **Claudio Canuto** *Convergence and Optimality for a class of Adaptive Virtual Element Methods.* Chair: Marco Verani

11:10-11:40 **Coffee Break**

11:40-12:05 **Simon Lemaire** *A low-order positive method for advection-diffusion on general meshes.* Chair: Emmanuil H. Georgoulis

12:05-12:30 **Zhaonan Dong** *A posteriori error estimates for discontinuous Galerkin methods on polygonal and polyhedral meshes.* Chair: Emmanuil H. Georgoulis

12:30-12:55 **Luca Formaggia** *Some reactive models for reactive flow in fractured porous media.* Chair: Emmanuil H. Georgoulis

12:55-14:30 **Lunch Break**

14:30-15:15 **Susanne C. Brenner** *Some Applications of the VEM Methodology.* Chair: Ilaria Perugia

15:15-16:00 **Alexander Ern** *Hybrid high-order methods for the biharmonic problem.* Chair: Ilaria Perugia

16:00-16:25 **N. Sukumar** *Scaled boundary cubature scheme in higher dimensions: Integration over polytopes and curved regions.* Chair: Ilaria Perugia

16:25-17:00 **Coffee Break**

17:00-17:25 **D. Prada** *A new fully p-robust domain decomposition preconditioner.* Chair: Alessandro Russo

17:25-17:50 **Stefano Scialó** *Orthogonal basis for Mixed Virtual Elements.* Chair: Alessandro Russo

**Tuesday 13 December**

9:00-9:45 **Ilaria Perugia** *Space-time virtual elements for the heat equation.* Chair: Paola F. Antonietti

9:45-10:30 **Emmanuil H. Georgoulis** *A posteriori error control of polytopic space-time discontinuous Galerkin methods for the Allen-Cahn problem.* Chair: Paola F. Antonietti

10:30-11:00 **Coffee Break**

11:00-11:25 **Eun-Jae Park** *Polygonal Staggered DG methods.* Chair: Alexander Ern

11:25-11:50 **Daniel Castañón-Quiroz** *A pressure-robust HHO method for the solution of the incompressible Navier–Stokes equations on general meshes* Chair: Alexander Ern

11:50-12:15 **Lina Zhao** *A pressure-robust staggered DG method for Brinkman problem with applications to coupled flow and transport.* Chair: Alexander Ern

12:15-13:00 **Stefano Bonetti**, **Mattia Corti**, **André Harnist**, **Liam Yemm**, **Julian Moatti**, *Lighting talks.* Chair: Franco Dassi

13:00-14:30 **Lunch Break**

14:30-15:15 **Bernardo Cockburn** *Variational principles for HDG methods.* Chair: Daniele Di Pietro

15:15-15:40 **Géraldine Pichot** *An agglomeration-based HHO method for flow simulations in Discrete Fracture Networks.* Chair: Daniele Di Pietro

15:40-16:05 **David Mora** *Stream Virtual Element Methods for the Brinkman Equations.* Chair: Daniele Di Pietro

16:05-16:30 **Omar Duran** *A mixed-dimensional formulation for fracture mechanics based on the linear theory of the Cosserat continuum.* Chair: Daniele Di Pietro

16:30-17:00 **Coffee Break**

17:00-17:25 **Stefano Berrone** *Stabilization free Virtual Element Method.* Chair: Susanne C. Brenner

17:25-17:50 **Silvia Bertoluzza** *The Virtual Element Method on image based approximate domains.* Chair: Susanne C. Brenner

**Wednesday 14 December**

9:00-9:45 **Peter Wriggers** *The Virtual Element Method for Computational Homogenization.* Chair: Lourenço Beirão da Veiga

9:45-10:30 **Sergio Gómez Alice Hodson Enrico Manuzzi Michele Visinoni Dibyendu Adak** *Lightning talks.* Chair: Lourenço Beirão da Veiga

10:30-11:00 **Coffee Break**

11:00-11:25 **Carlo Lovadina** *Some Virtual Element Methods for linear Elasticity Problems.* Chair: Claudio Canuto

11:25-11:50 **Carsten Carstensen** *Nonconforming virtual elements for the biharmonic equation with Morley degrees of freedom on polygonal meshes.* Chair: Claudio Canuto

11:50-12:15 **Fadi Aldakheel** *Virtual Elements for computational anisotropic crystal plasticity.* Chair: Claudio Canuto

12:15-13:00 **Elena Bachini Francesca Marcon Tommaso Sorgente Harry Wells** *Lightning talks.* Chair: Michele Botti

13:00-14:30 **Lunch Break**

14:30-14:55 **Giuseppe Vacca** *Divergence-free Virtual Elements for fluid dynamics.* Chair: N. Sukumar

14:55-15:20 **Massimo Frittelli** *The virtual element method for bulk-surface PDEs and its application to battery modeling.* Chair: N. Sukumar

15:20 - 15:35 **Closing**

# Abstracts

## Virtual element method for nondivergence form elliptic equations.

Normal talk

Andrea Cangiani

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Problems in nondivergence form arise from a wide range of applications such as stochastic optimal control, nonlinear elasticity, and image processing. Their numerical discretisation is notoriously challenging due to the lack of notion of weak solutions based on variational principles. We consider the model problem of nondivergence form elliptic equations arising from the linearization of fully nonlinear equations, such as the Hamilton-Jacobi-Bellman equation. The model problem is well-posed under the Cordès conditions thanks to the Miranda-Talenti inequality, bounding the $L^2$-norm of the Hessian of $H^2$ functions with zero trace on convex domains by their laplacian. We exploit the availability of $C^1$-conforming Virtual Element spaces to design a VEM based on an equivalent variational problem whose stability follows directly from the availability of the Miranda-Talenti inequality. The resulting VEM is proven to converge optimally in the $H^2$-norm, as confirmed by numerical experiments performed using the DUNE-VEM package. Joint work with Guillaume Bonnet and Ricardo H. Nochetto.

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## Convergence and Optimality for a class of Adaptive Virtual Element Methods

Plenary talk

Claudio Canuto

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We focus on non-conforming meshes obtained by successive newest vertex bisections, without completion (i.e., without removal of the newly added hanging nodes). For such meshes, which we see as made of particular Virtual Elements, we show that in the a posteriori error estimator proposed by Cangiani et al (2017) for VEM discretizations, the stabilization term can be removed, provided any chain of recursively created hanging nodes has uniformly bounded length. Based on this result, we prove the convergence of a 2-loop adaptive algorithm via a contraction argument. In addition, we introduce approximation classes relative to VEM spaces, and we investigate the complexity of the proposed algorithm, proving its quasi-optimality

with respect to these classes. This is a joint work with L. Beirao da Veiga, R.H. Nochetto, G. Vacca and M.Verani.

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Normal talk

## A low-order positive method for advection-diffusion on general meshes

Simon Lemaire

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We introduce an hybrid finite volume (HFV) method for time-dependent advection-diffusion, in the case when the continuous solutions are positive. The advection-diffusion equation is here thought as a building block for more complicated models satisfying positivity (or min/max) principles. The HFV scheme is based on a nonlinear rewriting of the model at hand, and is proved to be positivity-preserving. The scheme can handle general polytopal meshes, and reproduces at the discrete level an entropy-dissipation principle. Leveraging the latter entropy-dissipation relation, we prove the existence of (positive) solutions to the scheme. We also prove, in the context of boundary-driven thermal relaxation, that the discrete solutions converge exponentially fast in time towards the HFV interpolate of the thermal equilibrium. We finally assess the behaviour of the method in representative situations, and compare our results with a standard HFV scheme. This work has been realized in collaboration with C. Chainais-Hillairet (Univ. Lille, Inria), M. Herda (Inria, Univ. Lille), and J. Moatti (Inria, Univ. Lille).

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Normal talk

## A posteriori error estimates for discontinuous Galerkin methods on polygonal and polyhedral meshes

Zhaonan Dong

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We present a new residual-type energy-norm a posteriori error analysis for interior penalty discontinuous Galerkin (dG) methods for linear elliptic problems. The new error bounds are also applicable to dG methods on meshes consisting of elements with very general polygonal/polyhedral shapes. The case of simplicial and/or box-type elements is included in the analysis as a special case. In particular, for the upper bounds, an arbitrary number of very small faces are allowed on each polygonal/polyhedral element, as long as certain mild shape regularity assumptions are satisfied. As a corollary, the present analysis generalizes known a posteriori error bounds for dG methods, allowing in particular for meshes with an arbitrary number of irregular hanging nodes per element. The proof hinges on a new conforming recovery strategy in conjunction with a Helmholtz decomposition formula. The resulting a posteriori error bound involves jumps on the tangential derivatives along elemental faces. Local lower bounds are also proven for a number of practical cases. Numerical experiments are also presented, highlighting the practical value of the derived a posteriori error bounds as error estimators.

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## Some reactive models for reactive flow in fractured porous media

Normal talk

Luca Formaggia

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We give an overview of recent research activities in the compgeo@mox group of the Mox Laboratory of the Department of Mathematics of Politecnico di Milano, on the mathematical modelling simulation of reactive flow, in particular dissolution/precipitation phenomena in fractured porous media with an hybrid dimensional discretization scheme.

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## Some Applications of the VEM Methodology

Plenary talk

Susanne C. Brenner

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The applications of virtual elements to computational mechanics have met with many successes in recent years. In this talk I will present some applications of the virtual element methodology in other directions. It is based on joint work with Li-yeng Sung and Zhiyu.

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## Hybrid high-order methods for the biharmonic problem

Plenary talk

Alexandre Ern

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The talk starts with a gentle introduction to the devising and analysis of hybrid high-order (HHO) methods for the Poisson model problem. Then, we address the biharmonic problem and compare the proposed HHO methods to the literature, in particular to weak Galerkin methods. Finally, we briefly discuss how the error analysis can be carried out in the case of an exact solution with low regularity. This is joint work with Z. Dong.

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Normal talk

# Scaled boundary cubature scheme in higher dimensions: Integration over polytopes and curved regions

N. Sukumar

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In this talk, we introduce the scaled boundary cubature (SBC) scheme for accurate and efficient integration of functions over polytopes bounded by affine faces and curved surfaces. We pose the problem of numerical integration over a polytope in $\mathbb{R}^d$ as the decomposition of it into a sum of polytopal volumes where each polytope is formed by a point in $\mathbb{R}^d$ and a face of the polytope that is in $\mathbb{R}^{d-1}$. To this end, a nested convex combination (multilinear) map is constructed, so that integration is converted to that over the hypercube in which tensor-product univariate Gauss quadrature is used. The use of a multilinear map ensures that the numerical integration of polynomials can be exactly computed over arbitrary (convex and nonconvex) polytopes. For three-dimensional domains with oriented boundaries that are represented using tensor-product polynomial B-spline surfaces, rational Bezier triangles, or NURBS surfaces, integration using SBC requires evaluating the surface representation and its covariant basis, in addition to the integrand. Additionally, for star-convex domains, a tensor-product cubature rule with positive weights and integration points in the interior of the domain is obtained. The SBC scheme can also be applied to efficiently integrate point singularities (two-dimensional and three-dimensional domains) and line singularities (three-dimensional domains). If the integrand is homogeneous, we show that this new method reduces to the homogeneous numerical integration scheme; however, the SBC scheme is more versatile since it is equally applicable to homogeneous functions, polynomials as well as nonpolynomial functions. Applications of the SBC method to several benchmark problems will be presented to demonstrate its broad applicability and superior performance when compared to existing methods for numerical integration. This is joint work with Eric Chin at Lawrence Livermore National Laboratory.

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Normal talk

# A new fully p-robust domain decomposition preconditioner

Daniele Prada

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We build and analyze a new domain decomposition preconditioner for elliptic problems discretized by either the finite element method (FEM) or the virtual element method (VEM). We prove condition number bounds with possibly a weak dependence on the approximation order $p$. These bounds are also independent of jumps in the diffusion coefficients.

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## Orthogonal basis for Mixed Virtual Elements

Normal talk

Stefano Scialó

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The use of orthonormal polynomial basis proved to be an effective strategy to prevent ill-conditioning of the stiffness matrix in Virtual Elements in primal form, for high order approximations in presence of badly shaped elements. Such strategy is extended here to the mixed form of Virtual Elements. Different choices for the definition of the new orthonormal bases are presented along with some numerical results on poorly shaped elements arising from the meshing of domains with immersed interfaces.

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## Space-time virtual elements for the heat equation

Plenary talk

Ilaria Perugia

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Space-time methods, as opposed to time-marching schemes, are based on variational formulations of the considered time-dependent problems in both space and time. Advantages of this monolithic approach are that high-order approximations both in space and time are simple to obtain, simultaneous local refinement in space and time is possible, and the numerical solution is available at all times. In this talk, we present a space-time virtual element method for the approximation of the heat equation, which generalizes the Petrov-Galerkin finite element method introduced by O. Steinbach in 2015. The considered meshes are tensor products of polytopic meshes in space and interval partitions in time. Local test and trial functions are defined as solutions to a heat problem with polynomial data. Global approximation spaces are constructed in a nonconforming fashion. This allows for an analysis setting and an implementation strategy, which are independent of the spatial dimension. As typical of the virtual element framework, the basis functions are not known in closed form. The method is therefore defined in terms of degrees of freedom only, with the help of suitable local projections onto underlying space-time polynomial spaces. Theoretical results, as well as their numerical validation, will be discussed. These results have been obtained in collaboration with Sergio Gómez, Lorenzo Mascotto, and Andrea Moiola.

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Plenary talk

## A posteriori error control of polytopic space-time discontinuous Galerkin methods for the Allen-Cahn problem

Emmanuil H. Georgoulis

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We are concerned with the proof of a posteriori error bounds for fully- discrete Galerkin approximations of the Allen-Cahn equation in two and three spatial dimensions. We begin by discussing the case of the backward Euler method combined with conforming finite elements on standard meshes in space, before continuing with the case of space-time discontinuous Galerkin methods on very general, polytopic, prismatic meshes. For both methods, we prove conditional type a posteriori error estimates in the $L^4(L^4)$-norm that depend polynomially upon the inverse of the interface length $\varepsilon$. The derivation relies on the availability of a spectral estimate for the linearized Allen-Cahn operator about the approximating solution in conjunction with a continuation argument and a variant of the elliptic reconstruction. The new analysis also appears to improve variants of known a posteriori error bounds in $L^2(H^1)$, $L^\infty(L^2)$-norms in certain regimes. The presentation is based on joint work with K. Chrysafinos (NTU Athens), Z. Dong (INRIA, Paris) and D. Plaka (NTU Athens).

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Normal talk

## Polygonal Staggered DG methods

Eun-Jae Park

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In this talk, we first present the staggered discontinuous Galerkin (SDG) method on general meshes for the Darcy flow. Then, consider arbitrary polynomial order, pressure-robust SDG method for the stationary Navier-Stokes equations. The exact divergence-free condition for the velocity is satisfied thanks to the carefully designed finite element spaces. An edge-wise stabilization for the nonlinear convective term earning non-negativity is proposed. The optimal convergence estimates for all the variables in $L^2$ norm are proved. Also, assuming that the rotational part of the forcing term is small enough, we are able to prove that the velocity error is independent of the Reynolds number and of the pressure. Furthermore, superconvergence can be achieved for the velocity error under a suitable projection. Numerical experiments are provided to validate the theoretical findings and demonstrate the performances of the proposed method. This is joint work with Eric Chung, Dohyun Kim, and Lina Zhao.

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## A pressure-robust HHO method for the solution of the incompressible Navier–Stokes equations on general meshes

Normal talk

Daniel Castanon-Quiroz

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In this work we introduce and analyze a novel pressure-robust Hybrid High-Order method for the steady incompressible Navier–Stokes equations on general meshes. The proposed method supports arbitrary approximation orders, and is (relatively) inexpensive thanks to the possibility of statically condensing a subset of the unknowns at each nonlinear iteration. For regular solutions and under a standard data smallness assumption, we prove a pressure-independent energy error estimate on the velocity of order $(k+1)$. More precisely, when polynomials of degree $k \geq 0$ at mesh elements and faces are used, this quantity is proved to converge as $h^{k+1}$ (with $h$ denoting the meshsize). The proposed method is a direct extension of a previous work done by the same authors using simplicial meshes. Numerical results are presented to support the theoretical analysis.

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## A pressure-robust staggered DG method for Brinkman problem with applications to coupled flow and transport

Normal talk

Lina Zhao

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In this talk, I will first present a pressure-robust staggered DG method for the Brinkman problem. The scheme is free of stabilization and supports arbitrary shapes of polygon. The devising of the method hinges on a carefully balanced finite element pair; indeed, the tangential continuity of velocity is relaxed. Optimal convergence for all the variables is achieved. Then I will extend the scheme to solve coupled flow and transport, where the flow is governed by the coupled Brinkman-Darcy equations. Staggered DG method and mixed finite element method are judiciously combined to yield a strongly conservative scheme, which is particularly important for practical applications. Several numerical experiments will be presented to verify the performance of the proposed scheme.

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Lightning talk

# Discontinuos Galerkin approximation of the fully-coupled thermo-poroelastic problem

Stefano Bonetti

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Poroelasticity inspects the interaction among fluid flow and elastic deformations within a porous medium. In several applications in the context of environmental sustainability, such as geothermal energy production and CO2 sequestration, temperature plays a key role in the description of the physical phenomena. Thus, in order to correctly describe these geological processes, the differential problem should also take into account the influence of the temperature, leading to a fully-coupled thermo-poroelastic (TPE) system of equations [2,3]. In the framework of geosciences applications, the subsoil is modelled as a fully-saturated poroelastic material under the additional assumptions of small deformations and quasi-static regime. We present and analyze a discontinuous Galerkin method for the numerical modelling of the non-linear fully-coupled thermo-poroelastic problem. For the spatial discretization, we design a high-order discontinuous Galerkin method on polygonal and polyhedral grids based on a novel four-field formulation of the problem [4]. To handle the non-linear convective transport term in the energy conservation equation we adopt a fixed-point linearization strategy and different linearizations are examined. We perform a robust stability analysis for the linearized semi-discrete problem under mild requirements on the problem data. A priori *hp*-version error estimates in suitable energy norms are also derived. A complete set of numerical simulations is presented in order to validate the theoretical analysis, to inspect numerically the robustness properties, and to test the capability of the proposed method in a practical scenario inspired by a geothermal problem.

[1] P.F. Antonietti, S. Bonetti, M. Botti, "Discontinuos Galerkin approximation of the fully-coupled thermo-poroelastic problem", arxiv.org/abs/2205.04262 (2022)
[2] M.K. Brun, E. Ahmed, J.M. Nordbotten, F.A. Radu, "Well-posedness of the fully coupled quasi-static thermo-poroelastic equations with nonlinear convective transport", Journal of Mathematical Analysis and Applications, 471 (1), 239-266 (2019)
[3] M.K. Brun, I. Berre, J.M. Nordbotten, F.A. Radu, "Upscaling of the coupling of hydrome-chanical and thermal processes in a quasi-static poroelastic medium", Transport in Porous Media, 124(1), 137–158 (2018)
[4] R. Oyarzúa, R. Ruiz-Baier, "Locking-free finite element methods for poroelasticity", SIAM Journal on Numerical Analysis, 54(5), 2951–2973 (2016)

## Numerical Modelling of the Brain Poromechanics by High-Order Discontinuous Galerkin Methods

Lightning talk

Mattia Corti

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The Multiple-Network Poroelastic Theory (MPET) model can comprehensively describe functional changes in the brain considering multiple scales of fluids. Indeed, to model blood perfusion, it is essential to separate the vascular network into its fundamental components (arteries, capillaries and veins). Moreover, in brain physiology, the cerebrospinal fluid (CSF) has an important role in waste clearance [2]. We introduce and analyze a discontinuous Galerkin method for the numerical modelling of the equations of Multiple-Network Poroelastic Theory (MPET) in the dynamic formulation [1]. Concerning the spatial discretization, we employ a high-order discontinuous Galerkin method on polygonal and polyhedral grids and we derive stability and a priori error estimates. The temporal discretization is based on a coupling between a Newmark $\beta$-method for the momentum equation and a $\theta$-method for the pressure equations. We present some verification numerical tests, we perform a convergence analysis using an agglomerated mesh of a geometry of a brain slice. Finally, we present a simulation in a three dimensional patient-specific brain reconstructed from magnetic resonance images.

[1] M. Corti, P. F. Antonietti, L. Dede', and A. M. Quarteroni. Numerical modelling of the brain poromechanics by high-order discontinuous Galerkin methods. arXiv/abs/2210.02272, 2022.

[2] B. Tully and Y. Ventikos. Cerebral water transport using multiple-network poroelastic theory: Application to normal pressure hydrocephalus. Journal of Fluid Mechanics, 667:188–215, 2011

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## A Hybrid High-Order method for non-Newtonian fluids with power-like convective behaviour

Lightning talk

André Harnist

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The aim of this talk is to present a Hybrid High-Order (HHO) method for generalized Navier-Stokes equations adapted, not only to non-Newtonian fluids, but also to fluids with non-classical convective behaviour. This generalization is achieved by inserting viscosity and convection laws which satisfy certain assumptions. These laws are associated with possibly different Sobolev exponents, namely $r \in (1, \infty)$ for the viscosity law and $s \in (1, \infty)$ for the convection law. After providing a novel weak formulation of the continuous problem, we design a HHO scheme based on this weak formulation. The main results will be briefly presented, including conditions for well-posedness, convergence for general data, and error estimates for small data. These results highlight how a subtle interplay between the exponents $r$ and $s$ can affect the analysis of the continuous and discrete problems. The final part of the presentation will be dedicated to illustrating the method with a well-known problem in fluid mechanics, the lid-driven cavity flow. We will observe how the exponents $r$ and $s$ influence the viscous and convective behaviours of the fluid, respectively.

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Lightning talk

## Hybrid High-Order methods on curved meshes

Liam Yemm

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Hybrid high-order methods are a modern numerical framework for the approximation of elliptic PDEs. We present here an extension of the hybrid high-order method to meshes possessing curved edges/faces. Such an extension allows us to enforce boundary conditions exactly on curved domains, and capture curved geometries that appear internally in the domain e.g. discontinuities in a diffusion coefficient. We show the method to be stable and consistent on such meshes and provide numerical examples in two dimensions.

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Lightning talk

## A structure preserving hybrid finite volume scheme for semi-conductor models with magnetic fields on general meshes

Julian Moatti

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We are interested in the numerical approximation of anisotropic drift-diffusion systems, consisting of two anisotropic advection-diffusion equations coupled to a Poisson equation. Such systems describe the behaviour of semiconductor devices when immersed in an exterior magnetic field. Our method is based on a nonlinear hybrid finite volume (HFV) scheme for the advection-diffusion equations, coupled to a standard HFV scheme for the Poisson equation. It applies to general meshes and relies on the conservation at the discrete level of an entropy structure, which mimics the behaviour of the continuous system. Using these properties, we show the existence of solutions to the scheme and ensure bounds on the carrier densities (especially, the densities are positive). Moreover, we establish the convergence of the discrete solutions towards a discrete thermal equilibrium as time tends to infinity. We finally confirm our theoretical results and validate the robustness of the scheme with some numerical experiments.

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## Variational principles for HDG methods

Plenary talk

Bernardo Cockburn

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In the framework of steady-state diffusion, we show that there are eight variational formulations which can be used to devise the Hybridizable Discontinuous Galerkin (HDG) methods. Four of them are associated to the minimization of the energy and the other four to the minimization of the complementary energy. We show how most well known finite element methods are limits of the HDG methods. We also show that the very same method can be obtained by using any of the eight above-mentioned formulations. In this case, the choice of the variational formulation amounts to choosing the way the method is implemented.

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## An agglomeration-based HHO method for flow simulations in Discrete Fracture Networks

Normal talk

Géraldine Pichot

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Over the past years, Inria and the University of Technology of Troyes (UTT) have developed a mesh generator, called MODFRAC, capable of meshing large scale Discrete Fracture Networks (DFNs) containing more than one million of fractures. The mesh consists of triangles and is conforming to the intersections between fractures. This can be realized in a matching way with nodes matching at the intersections or in a nonmatching way with nodes from each fracture that may differ. Recently, the Hybrid High Order (HHO) method has been successfully tested to solve flows in such large DFNs using matching and nonmatching meshes. To further reduce the number of DOFs, we now propose to perform mesh agglomeration that is to say to merge the triangles (in possibly non convex) polygons. In this talk, we will present an agglomeration strategy combined with HHO and show, on some examples, how it allows us to save computational resources. This is a joint work with Zhaonan Dong (Inria Paris).

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## Stream Virtual Element Methods for the Brinkman Equations

Normal talk

David Mora

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In this talk, we develop a virtual element discretization for the Brinkman equations formulated in terms of the stream-function of the velocity field. We write a variational formulation and propose a virtual element discretization of arbitrary order $k \geq 2$. The velocity is obtained as a postprocess from stream-function discrete and under standard assumptions on the computational domain we prove error estimates for the stream-function and velocity. Finally, we present some numerical results.

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Normal talk

## A mixed-dimensional formulation for fracture mechanics based on the linear theory of the Cosserat continuum

Omar Duran

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This research was inspired and intended as an extension to the mixed-dimensional poromechanical models for fractured media presented in [1]. There the classical continuum mechanics is considered in the context of finite and infinitesimal strain. We addressed fracture mechanics from a different perspective by considering the linear theory of the Cosserat (micropolar, asymmetric) continuum. We considered an hybrid high-order (HHO) method for the resulting set equations. Extra regularity can be achieved on the asymmetric component of the stress tensor and fracture tractions through the angular momentum balance defined on surfaces. We addressed the second-order formulation and present numerical results for the adopted model.

[1] W.-M. Boon and J.-M. Nordbotten. Mixed-dimensional poromechanical models of fractured porous media, arXiv:2112.05038 (2021)

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Normal talk

## The Virtual Element Method on image based approximate domains

Silvia Bertoluzza

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We analyze and validate the virtual element method combined with a projection approach similar to the one proposed by Bramble, Dupont and Thomée, to solve problems on domains with curved boundaries approximated by polytopal domains obtained as the union of squared/cubic elements out of a uniform structured mesh, such as the one that naturally arises when the domain is issued from imaging. In such a framework, resorting to the use of polygonal element allows to satisfy the assumptions required for the stability of the projection method, thus allowing to fully exploit the potential of higher order methods, which makes the resulting approach an effective alternative to other approaches. This is a joint work with M. Montardini, M. Pennacchio and D. Prada.

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## Stabilization free Virtual Element Method

Normal talk

Stefano Berrone

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In this talk, we introduce the idea of an Enlarged Enhanced Virtual Element basis to avoid the non-conforming stabilization term in the formulation of reaction-convection-diffusion problems. In many applications, the non-conforming stabilization term can introduce a non-physical perturbation of the model. The talk will cover some theoretical aspects, implementation and some practical examples.

[1] Stefano Berrone, Andrea Borio, Francesca Marcon, Lowest order stabilization free Virtual Element Method for the Poisson equation, arXiv:2103.16896v3, 2021.

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## The Virtual Element Method for Computational Homogenization

Plenary talk

Peter Wriggers

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This work presents a study on the computational homogenization in solid mechanics using the Virtual Element Method. VEM has a great potential for the homogenization of the physical properties of heterogeneous polycrystalline microstructures with isotropic and anisotropic grains. The flexibility in element shapes can be exploited for creating virtual element mesh with a significant lower number of degrees of freedom compared to finite element meshes, while maintaining a high accuracy.

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## High-order interpolatory/quasi-interpolatory serendipity virtual element method for semilinear parabolic problems

Lightning talk

Sergio Gómez

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An efficient method for the numerical approximation of a general class of two dimensional semilinear parabolic problems on polygonal meshes is presented. The proposed approach takes advantage of the properties of the serendipity version of the virtual element method (VEM), which not only reduces the number of degrees of freedom compared to the classical VEM, but also allows for the introduction of an interpolatory or quasi-interpolatory approximation of the nonlinear term that is computable from the degrees of freedom of the discrete solution with a low computational cost, thus significantly improving the efficiency of the method. The accuracy and efficiency of the proposed method when combined with a second order Strang operator splitting time discretization is illustrated with numerical experiments, with approximations up to order 6.

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Lightning talk

## Virtual element methods for fourth-order problems

Alice Hodson

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In this talk we discuss virtual element methods for a wide range of fourth-order problems. The construction of even a lowest order $C^1$ conforming space is not straightforward within the standard finite element setting and higher order non-conforming spaces suitable for fourth-order problems are also not readily available. Consequently, many software packages only provide the lowest order Morley element for discretising fourth-order problems without requiring the use of splitting techniques. In this talk, we follow the approach of defining a hierarchy of projection operators for the necessary derivatives with the starting point being a constraint least squares problem. We show that by defining the projection operators without using the underlying variational problem, we can directly apply our method to both variable coefficient and nonlinear problems. Another major advantage of this approach is that it can also be integrated more easily into existing finite element software frameworks. We demonstrate this within the open source DUNE software package.

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Lightning talk

## Machine Learning-based Refinement and Agglomeration of Polytopal Grids

Enrico Manuzzi

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We propose new strategies based on Machine Learning techniques to handle polytopal grid refinement and agglomeration. Concerning mesh refinement, we employ Convolutional Neural Networks to classify the “shape” of an element so that “ad-hoc” refinement criteria can be defined. To improve robustness, we employ the k-means clustering algorithm to refine mesh elements with no particular structure. Concerning mesh agglomeration, we employ Graph Neural Network to partition the connectivity graph of mesh elements, processing simultanously also geometrical features. The effectiveness of the proposed strategies is demonstrated in terms of fast online inference, preservation of mesh quality, low computational complexity and high performance when applied to the Virtual Element Methods, Polytopal Discontinuous Galerkin methods and multigrid solvers.

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## Hybridization procedure of the Virtual Element Method for elasticity problems

Lightning talk

Michele Visinoni

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We present the hybridization strategy proposed in [Arnold, Brezzi, 1985, M2AN] to the Virtual Element Method for linear elasticity problems based on the Hellinger-Reissner variational principle. In this talk, we detail the fundamental idea behind this technology, and we show the advantages which led to solving the resulting linear system in a very efficient way and designing a better approximation of the displacement field using a suitable post-processing procedure. A series of numerical experiments are provided to show the performance of our proposed approach.

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## The nonconforming virtual element method for Oseen's equation using a stream-function formulation

Lightning talk

Dibyendu Adak

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We approximate the solution of the stream function formulation of the Oseen equations on general domains by designing a nonconforming Morley-type virtual element method. Under a suitable assumption on the continuous problem's coefficients, the discrete scheme is well-posed. By introducing an enriching operator, we derive an a priori estimate of the error in a discrete $H^2$ norm. By post-processing the discrete stream function, we compute the discrete velocity and vorticity fields. Furthermore, we recover an approximate pressure field by solving a Stokes-like problem in a nonconforming Crouzeix-Raviart-type virtual element space that is in a Stokes-complex relation with the Morley-type space of the virtual element approximation. Finally, we confirm our theoretical estimates by solving benchmark problems that include a convex and a nonconvex domain.

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## Some Virtual Element Methods for linear Elasticity Problems

Normal talk

Carlo Lovadina

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In this talk we consider the Virtual Element Method applied to linear elasticity problems, focusing on mixed (and hybrid) schemes. In particular, we will present Virtual Elements based on the Hellinger-Reissner functional, but in a 2D and in a 3D setting. Those methods, for which a rigorous stability and convergence analysis is available, represent a valid alternative to Finite Element schemes, as demonstrated by several numerical tests.

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Normal talk

# Nonconforming virtual elements for the biharmonic equation with Morley degrees of freedom on polygonal meshes

Carsten Carstensen

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The lowest-order nonconforming virtual element extends the Morley triangular element to polygons for the approximation of the weak solution $u \in V := H_0^2(\Omega)$ to the biharmonic equation. Two nonconforming virtual element spaces have been introduced in [1,4] for $H^3$ regular solutions, while a medius analysis in [3] allows minimal regularity. In this talk, we will discuss an abstract framework of [2] with two hypotheses (H1)-(H2) for a unified stability and *a priori* error analysis of at least two different discrete spaces (even a mixture of those). A smoother $J$ allows rough source terms $F \in V^* = H^{-2}(\Omega)$. The *a priori* and *a posteriori* error analysis circumvents any trace of second derivatives by a computable conforming companion operator $J : V_h \to V$ from the nonconforming virtual element space $V_h$. The operator $J$ is a right-inverse of the interpolation operator and leads to optimal error estimates in piecewise Sobolev norms without any additional regularity assumptions on $u \in V$. As a smoother the companion operator modifies the discrete right-hand side and then allows a quasi-best approximation. An explicit residual-based *a posteriori* error estimator is reliable and efficient up to data oscillations. Numerical examples display the predicted empirical convergence rates for uniform and optimal convergence rates for adaptive mesh-refinement. The presentation is based on [2] and ongoing work with R. Khot.

[1] P. F. Antonietti, G. Manzini, and M. Verani, The fully nonconforming virtual element method for biharmonic problems, Math. Models Methods Appl. Sci. 28 (2018), no. 02, 387–407.

[2] C. Carstensen, R. Khot, and A. K. Pani, Nonconforming virtual elements for the biharmonic equation with Morley degrees of freedom on polygonal meshes, arXiv:2205.08764 (2022).

[3] J. Huang and Y. Yu, A medius error analysis for nonconforming virtual element methods for Poisson and biharmonic equations, J. Comput. Appl. Math. 386 (2021), no. 113229, 21.

[4] J. Zhao, B. Zhang, S. Chen, and S. Mao, The Morley-type virtual element for plate bending problems, J. Sci. Comput. 76 (2018), no. 1, 610–629.

## Virtual Elements for computational anisotropic crystal plasticity

Normal talk

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In this presentation, the Virtual Element Method (VEM) with a linear ansatz is applied to a computational crystal plasticity framework in a micro-structural environment. Furthermore, a simple anisotropic energetic contribution, based on invariant formulations of tensorial deformation measures and structural tensors, is presented for the cubic elastic anisotropy of the underlying crystal structure. The anisotropic elastic formulation recovers the elasticity tensor structure of a cubic material within the limit of small deformations. We propose a new stabilization degradation formulation that is purely based on the dissipative response of the problem. Representative examples illustrate the robustness and performance of VEM with regard to locking phenomena in the crystal plasticity framework when benchmarked against the solutions of classical finite element approaches. Further examples investigate the performance and current limitations of VEM within a crystal plasticity framework when being applied to heterogeneous microstructures for both, structured element topology as well as flexible element topology.

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## Numerical solution of surface PDEs via a geometrically Intrinsic Virtual Element Method

Lightning talk

Elena Bachini

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We present a geometrically intrinsic formulation of the arbitrary-order Virtual Element Method (VEM) on polygonal cells for the numerical solution of surface elliptic PDEs. The equation is first written in covariant form using an appropriate local reference system, and the knowledge of the local parametrization allows us to derive a two-dimensional VEM scheme, without any explicit approximation of the surface geometry. The final discretization presents a highly anisotropic character, depending on the geometric features of the surface. We show numerical results on triangular and polygonal meshes using manufactured solutions.

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Lightning talk

## Stabilization free virtual element methods: key points, comparisons and applications

Francesca Marcon

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The aim of this talk is to briefly present a new family of Virtual Element Methods (VEM), that we call Enlarged Enhancement Virtual Element Methods (E2VEM), designed to allow the definition of a coercive bilinear form that involves only polynomial projections, thus preserving the structure of the natural bilinear form. We compare the behaviour of standard VEM and E2VEM, with the focus on convection-dominated problems and on some elliptic test problems whose solution and diffusivity tensor are characterized by anisotropies. Results show that the possibility of a polynomial self-stabilized bilinear form, offered by E2VEM methods, can in some cases positively influence the magnitude of the error.

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Lightning talk

## Mesh Quality Indicators for the Virtual Element Method

Tommaso Sorgente

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We present the design of a mesh quality indicator that can predict the behavior of the Virtual Element Method on a given mesh family or a finite sequence of meshes. The indicator is designed to measure the violation of the mesh regularity assumptions that are normally considered in the convergence analysis. It is capable of foreseeing the performance of the VEM over very general polygonal and polyhedral meshes, including meshes containing non-convex and skewed elements.

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Lightning talk

## Developing Moving Mesh Virtual Element Methods

Harry Wells

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In this short talk I will present works on developing a moving mesh virtual element method. This will include a discussion on how to extend classical moving mesh finite element methods to a virtual element framework. Successfully developed methods will be presented along with numerical experiments and benchmark applications. This talk contains work done under the supervision of M.E. Hubbard (University of Nottingham) and A. Cangiani (SISSA) and in collaboration with A. Dedner (University of Warwick).

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## Divergence-free Virtual Elements for fluid dynamics

Normal talk

Giuseppe Vacca

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The present talk is both an introduction to the divergence-free Virtual Elements (VEMs) for the Navier-Stokes equation aiming at showing the main ideas of the method, and a brief look at some applications to the fluid-dynamic problems. In the first part of the talk we will describe the basics of the divergence-free VEMs for the Navier-Stokes equation and we will explore the main features and the advantages of the “divergence-free construction”. In the second part we will present the extension of the divergence-free VEM to more applied problems among the following: the convection dominated Oseen equation, the Boussinesq equation, the Magnetohydrodynamics equation, a fluid structure interaction problem.

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## The virtual element method for bulk-surface PDEs and its application to battery modeling

Normal talk

Massimo Frittelli

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We present the Bulk-Surface Virtual Element Method (BSVEM) for the spatial approximation of stationary and time-dependent coupled bulk-surface PDEs in three space dimensions. The method combines the polyhedral VEM for the bulk equations [Beirao da Veiga et al., CAMWA, 2017] and the surface Virtual Element Method (SVEM) for the surface equations [Frittelli et al., 2018, M2AN]. We provide a geometric error analysis of polyhedral meshes independent of the method. Then, we provide a full error analysis in the lowest order case that holds even in the presence of curved boundaries. We also show a higher order variant of the method, where curved faces take care of the geometric error. The method brings all the advantages of general polyhedral meshes into the context of bulk-surface PDEs. We show applications of the method to battery modelling.

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