Hybridization of VEM for Hellinger-Reissner elasticity problems

Michele Visinoni
University of Milan-Bicocca

a joint work with:
Franco Dassi (University of Milan-Bicocca)
Carlo Lovadina (University of Milan)

INdAM Workshop
“Polygonal methods for PDEs: theory and applications”
17-19 May 2021
Outline of the talk

1. Continuous problem
2. Motivation
3. Hybridization procedure and VEM
4. Numerical results
Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and let $f \in [L^2(\Omega)]^2$ be the loading term. We find $(\sigma, u)$ such that:

$$
\begin{cases}
- \text{div} \sigma = f & \text{in } \Omega \\
\sigma = C \varepsilon(u) & \text{in } \Omega \\
u = 0 & \text{in } \partial \Omega
\end{cases}
$$

where:

- $\varepsilon(u)$ is the symmetric gradient of $u$,
- $C$ is the elasticity four-order symmetric tensor.

The constitutive law:

$$
\sigma = C \varepsilon(u) := 2\mu \varepsilon(u) + \lambda \text{tr}(\varepsilon(u)) \text{Id}.
$$
The **mixed** variational formulation:

\[
\begin{cases}
\text{Find } (\sigma, u) \in \Sigma \times U \text{ such that} \\
a(\sigma, \tau) + b(\tau, u) = 0 \quad \forall \tau \in \Sigma \\
b(\sigma, v) = -(f, v) \quad \forall v \in U
\end{cases}
\]

where:

- \( U = [L^2(\Omega)]^2 \),
- \( \Sigma = \{ \tau \in H(\text{div}; \Omega) : \tau \text{ is symmetric} \} \).

The bilinear forms and the right-hand side are defined as follows:

- \( a(\sigma, \tau) := \int_{\Omega} C^{-1} \sigma : \tau \, d\Omega \).
- \( b(\tau, u) := \int_{\Omega} \text{div} \tau \cdot u \, d\Omega \).
- \( (f, v) := \int_{\Omega} f \cdot v \, d\Omega \).
Motivation

We consider the following low-order virtual element schemes:

- **E. Artioli, S. de Miranda, C. Lovadina, L. Patruno:**

- **F. Dassi, C. Lovadina, M. V.:**

**Goal:** Present the hybridization procedure for low-order VE schemes for both two and three-dimensional problems.
We consider the following low-order virtual element schemes:

- **E. Artioli, S. de Miranda, C. Lovadina, L. Patruno:**

- **F. Dassi, C. Lovadina, M. V.:**

**Goal:** Present the hybridization procedure for low-order VE schemes for both two and three-dimensional problems.
The hybridization procedure (Fraeijs De Veubeke, 1965) is a computational technique for mixed method which leads to solving symmetric and positive definite linear system instead of the original indefinite one.

This procedure is split into two different steps:

- the imposition of the stress $H(\text{div})$-conformity requirement through the introduction of suitable Lagrange multipliers;
- the application of the static condensation algorithm.

Exploiting the information derived from the Lagrange multipliers (Arnold, Brezzi, 1985), there is the possibility to reconstruct the discrete solution through post-processing, i.e., using non-conforming VEM (Ayuso, Lipnikov, Manzini, 2016).
The hybridization procedure (Fraeijs De Veubeke, 1965) is a computational technique for mixed method which leads to solving symmetric and positive definite linear system instead of the original indefinite one.

This procedure is split into two different steps:

- the imposition of the stress $H(\text{div})$-conformity requirement through the introduction of suitable Lagrange multipliers;
- the application of the static condensation algorithm.

Exploiting the information derived from the Lagrange multipliers (Arnold, Brezzi, 1985), there is the possibility to reconstruct the discrete solution through post-processing, i.e., using non-conforming VEM (Ayuso, Lipnikov, Manzini, 2016).
Local space $U_h(E)$

We define the local displacement space:

$$U_h(E) = \left\{ v_h \in \left[ L^2(E) \right]^2 : v_h \in RM(E) \right\},$$

where we define the space of the local infinitesimal rigid body motions as

$$RM(E) := \left\{ \mathbf{r}(\mathbf{x}) = \alpha + \omega (\mathbf{x} - \mathbf{x}_E)^\perp \text{ s.t. } \alpha \in \mathbb{R}^2, \omega \in \mathbb{R} \right\}.$$
Local space $\Sigma_h(E)$

We define the local stress space:

$$\Sigma_h(E) := \left\{ \tau_h \in H(\text{div}; E) : \exists \mathbf{w}^* \in \left[ H^1(E) \right]^2 \text{ s.t. } \tau_h = C\varepsilon(\mathbf{w}^*); \right\}$$

$$(\tau_h \mathbf{n})|_e \in R(e) \quad \forall e \in \partial E; \quad \text{div} \tau_h \in RM(E) \right\}.$$

For each $e \in \partial E$ we define

$$R(e) := \left\{ \mathbf{t}(s) = c \mathbf{t}_e + p_1(s) \mathbf{n}_e \quad \text{s.t. } c \in \mathbb{R}, \ p_1(s) \in \mathcal{P}_1(e) \right\},$$

where $\mathbf{n}_e$ is the outward normal to the edge $e$ and $\mathbf{t}_e$ is the vector tangent to the edge $e$. 

M. Visinoni et al.  VEM hybridization for HR problems  May 19th, 2021  8 / 18
Lagrange multipliers

Let $\mathcal{E}_h^I$, the set of the internal edges of $\mathcal{T}_h$.

We define the space of the Lagrange multipliers as follows:

$$\Lambda_h(\mathcal{E}_h^I) := \left\{ \mu_h \in \left[ L^2(\mathcal{E}_h^I) \right]^2 : \mu_h|_e \in R(e) \quad \forall e \in \mathcal{E}_h^I \right\},$$

To force such a continuity we consider the bilinear form

$$c_h(\cdot, \cdot) : \tilde{\Sigma}_h(\mathcal{T}_h) \times \Lambda_h(\mathcal{E}_h^I) \to \mathbb{R}$$

defined as:

$$c_h(\tau_h, \mu_h) := -\sum_{E \in \mathcal{T}_h} \int_{\partial E^I} \mu_h \cdot \tau_h \, n \, ds \quad \forall \tau_h \in \tilde{\Sigma}_h(\mathcal{T}_h), \ \forall \mu_h \in \Lambda_h(\mathcal{E}_h^I),$$

where $\partial E^I = \partial E \cap \mathcal{E}_h^I$. 

Virtual Element Method

Let $\mathcal{T}_h$ be a decomposition of $\Omega$ into polygonal elements $E$. We consider the following discrete scheme:

\[
\begin{cases}
\text{Find } (\sigma_h, u_h, \lambda_h) \in \tilde{\Sigma}_h(\mathcal{T}_h) \times U_h \times \Lambda_h(\mathcal{E}_h^I) \text{ such that } \\
a_h(\sigma_h, \tau_h) + b(\tau_h, u_h) + c_h(\tau_h, \lambda_h) = 0 \\
b(\sigma_h, v_h) = -(f, v_h) \\
c_h(\sigma_h, \mu_h) = 0
\end{cases}
\forall \tau_h \in \tilde{\Sigma}_h(\mathcal{T}_h), \\
\forall v \in U_h, \\
\forall \mu_h \in \Lambda_h(\mathcal{E}_h^I).
\]

where

\[
\tilde{\Sigma}_h(\mathcal{T}_h) = \left\{ \tau_h \in \left[ L^2(\Omega) \right]^{2 \times 2} : \tau_h|_E \in \Sigma_h(E) \quad \forall E \in \mathcal{T}_h \right\}
\]

\[
U_h = \left\{ v_h \in \left[ L^2(\Omega) \right]^2 : v_h|_E \in U_h(E) \quad \forall E \in \mathcal{T}_h \right\}
\]

\[
\Lambda_h(\mathcal{E}_h^I) = \left\{ \mu_h \in \left[ L^2(\mathcal{E}_h^I) \right]^2 : \mu_h|_e \in R(e) \quad \forall e \in \mathcal{E}_h^I \right\}
\]
Consider the following mesh assumptions for $\mathcal{T}_h$:

- $E$ is star-shaped with respect to a ball of radius $\geq \gamma h_E$,
- the distance between any two vertexes of $E$ is $\geq \gamma h_E$.

where $\gamma$ is a suitable positive constant.

**Theorem**

*Let the above mesh assumptions hold. Let $(\sigma, u) \in \Sigma \times U$ be the solution of the continuous Problem, and $(\sigma_h, u_h) \in \tilde{\Sigma}_h(\mathcal{T}_h) \times U_h$ the solution of the discrete formulation. Assuming $(\sigma, u)$ sufficiently regular, the following estimate holds true:*

$$||\sigma - \sigma_h||_{\Sigma} + ||u - u_h||_U \lesssim h.$$
Numerical results: test case 2D

Given $\Omega = [0, 1]^2$ the unit square, we consider the following analytical solution

$$u : = \begin{pmatrix}
0.5(\sin(2\pi x))^2 \sin(2\pi y) \cos(2\pi y) \\
-0.5(\sin(2\pi y))^2 \sin(2\pi x) \cos(2\pi x)
\end{pmatrix}.$$

The loading term $f$ is computed accordingly. For this problem we consider a homogeneous and isotropic material with Lamé coefficients $\lambda = 10^5$ and $\mu = 0.5$ (nearly incompressible material).
Numerical results: test case 3D

Give the unit cube $\Omega = [0, 1]^3$, we consider a 3D elastic problem with the following exact displacement solution and load term:

\[
\begin{align*}
    u_1 &= u_2 = u_3 = 10S(x, y, z) \\
    f_1 &= -10\pi^2((\lambda + \mu)\cos(\pi x)\sin(\pi y + \pi z) - (\lambda + 4\mu)S(x, y, z)) \\
    f_2 &= -10\pi^2((\lambda + \mu)\cos(\pi y)\sin(\pi x + \pi z) - (\lambda + 4\mu)S(x, y, z)) \\
    f_3 &= -10\pi^2((\lambda + \mu)\cos(\pi z)\sin(\pi x + \pi y) - (\lambda + 4\mu)S(x, y, z))
\end{align*}
\]

where $S(x, y, z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$. In this case, we consider a compressible material where the Lamé constants are $\lambda = 1$ and $\mu = 1$. 
### Numerical results: comparison of solving time

<table>
<thead>
<tr>
<th>Step</th>
<th>Cube Standard</th>
<th>Cube Hybrid</th>
<th>Cube Hybrid (%)</th>
<th>Tetra Standard</th>
<th>Tetra Hybrid</th>
<th>Tetra Hybrid (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.11</td>
<td>0.11</td>
<td>(38.02%)</td>
<td>0.01</td>
<td>0.11</td>
<td>(32.14%)</td>
</tr>
<tr>
<td>2</td>
<td>5.74</td>
<td>3.09</td>
<td>(82.06%)</td>
<td>3.80</td>
<td>2.28</td>
<td>(70.37%)</td>
</tr>
<tr>
<td>3</td>
<td>971.33</td>
<td>209.53</td>
<td>(97.15%)</td>
<td>568.12</td>
<td>284.78</td>
<td>(97.43%)</td>
</tr>
<tr>
<td>4</td>
<td>4178.47</td>
<td>903.67</td>
<td>(98.78%)</td>
<td>3393.64</td>
<td>1409.92</td>
<td>(98.33%)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step</th>
<th>CVT Standard</th>
<th>CVT Hybrid</th>
<th>CVT Hybrid (%)</th>
<th>Rand Standard</th>
<th>Rand Hybrid</th>
<th>Rand Hybrid (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.86</td>
<td>0.68</td>
<td>(63.22%)</td>
<td>1.22</td>
<td>0.91</td>
<td>(67.32%)</td>
</tr>
<tr>
<td>2</td>
<td>97.88</td>
<td>53.43</td>
<td>(94.88%)</td>
<td>161.21</td>
<td>72.13</td>
<td>(95.04%)</td>
</tr>
<tr>
<td>3</td>
<td>29062.80</td>
<td>6877.68</td>
<td>(99.56%)</td>
<td>32015.50</td>
<td>14565.00</td>
<td>(99.70%)</td>
</tr>
<tr>
<td>4</td>
<td>128626.00</td>
<td>41000.70</td>
<td>(99.86%)</td>
<td>172781.00</td>
<td>81037.80</td>
<td>(99.91%)</td>
</tr>
</tbody>
</table>
Figure: Post-processing. Convergence plots for the error $E_{u_h}$ and $E_{u_h^*}^0$ for test case 2D.
Figure: Post-processing. Convergence plots for the error $E_{u_h}^1$ for test case 2D.
Figure: Post-processing. Convergence plots for the error $E_{u_h}^0$ and $E_{u_h}^1$ for test case 3D.
F. Dassi, C. Lovadina, and M. Visinoni.
Hybridization of a virtual element method for linear elasticity problems.


Thank you for your attention