Scaled Boundary Cubature Scheme for Numerical Integration over Planar Regions with Affine and Curved Boundaries

INdAM Workshop: Polygonal Methods for PDES: Theory and Applications
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Eric B. Chin¹, N. Sukumar²

¹ Lawrence Livermore National Laboratory
² University of California, Davis

* E. B. Chin and NS, Comput Methods Appl Mech Eng (2021)
Numerical integration over polygons and polyhedra
Need for integration over polytopes and curved regions

- Virtual element method and DG on polytopes
- Polygonal and polyhedral finite element methods
- Extended finite element method (cracks, holes, interfaces)
- Domain decomposition/contact
- Fictitious domain, finite cell, cut-cell methods, cutFEM
Outline

- Scaled boundary (SB) parametrization
- Scaled boundary cubature (SBC) over planar regions
- Simplification for homogeneous functions
- Integration of weakly singular functions
- Numerical examples
- Conclusions and outlook
Scaled boundary (SB) parametrization

\[ \Omega \quad t \in [0,1] \]

\[ c_i(0) = v_i \]

\[ c_i(1) = v_{i+1} \]
Scaled boundary (SB) parametrization

\[ t \in [0, 1] \]
\[ c_i(0) = \mathbf{v}_i \]
\[ c_i(1) = \mathbf{v}_{i+1} \]

\[ \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_6 = \Omega \]
Scaled boundary (SB) parametrization

- SB parametrization \( \varphi : [0,1]^2 \rightarrow \mathcal{T} \)
  \[
  \varphi(\xi, t) = x_0 + \xi (c(t) - x_0)
  \]
Scaled boundary (SB) parametrization

\[ \mathbf{x}_0 + \xi (c(t) - x_0) \]

- SB parametrization \( \varphi : [0,1]^2 \rightarrow \mathcal{T} \)

- \( \varphi(\xi, t) = x_0 + \xi (c(t) - x_0) \)

Scaled boundary cubature (SBC) over planar regions

- Given \( f : \mathcal{T} \to \mathbb{R} \), find

\[
I = \iint_{\mathcal{T}} f(x) \, dx
\]

- Introduce SB parametrization \( \varphi : [0,1]^2 \to \mathcal{T} \)

\[
\varphi(\xi, t) = x_0 + \xi (c(t) - x_0)
\]

- Now,

\[
I = \iint_{\mathcal{T}} f(x) \, dx = \iiint_{0}^{1} (f \circ \varphi)(\xi, t) \det \left( \nabla \varphi(\xi, t) \right) d\xi dt
\]

\[
= \iiint_{0}^{1} (f \circ \varphi)(\xi, t) \xi (c(t) - x_0) \cdot c'(t) \, d\xi dt
\]

where \( c'(t) \) is the vector in the normal direction, i.e., \( c'(t) \) rotated 90 degrees clockwise
Scaled boundary cubature (SBC) over planar regions

\[ I = \int_{\mathcal{F}} f(x) \, dx = \int_{0}^{1} \left( f \circ \varphi \right)(\xi, t) \xi (c(t) - x_0) \cdot c'_{\perp}(t) \, d\xi dt \]

- If \( f \) is a degree-\( q \) polynomial and \( c(t) \) is a degree-\( p \) polynomial curve, integrand is polynomial
  - Degree-\( (q+1) \) in \( \xi \)
  - Degree-\( [(q+2)p-1] \) in \( t \)
  - Exact integration using Gauss quadrature

- If \( c(t) \) is affine \( (c(t) - x_0) \cdot c'_{\perp}(t) = 2A_{\mathcal{F}} \)

\[ I = \int_{\mathcal{F}} f(x) \, dx = 2A_{\mathcal{F}} \int_{0}^{1} (f \circ \varphi)(\xi, t) \xi \, d\xi dt \]
Scaled boundary cubature (SBC) over planar regions

- Integrals over $\Omega$ can be decomposed to integrals over
  - Triangles (affine edge)
  - Curved triangles (curved edge)

$$I = \int_{\Omega} f(x) \, dx$$

$$= \sum_{i=1}^{3} 2A_{\mathcal{T}_i} \int_{0}^{1} \int_{0}^{1} (f \circ \varphi)(\xi, t) \xi \, d\xi \, dt$$

$$+ \sum_{i=4}^{6} \int_{0}^{1} \int_{0}^{1} (f \circ \varphi)(\xi, t) \xi (c_i(t) - x_0) \cdot c_i^\perp(t) \, d\xi \, dt$$
Simplification for homogeneous functions

- Assume \( f(x) \) is \( q \)-homogeneous
  - Choose \( x_0 = 0 \), such that \( \varphi(\xi, t) = \xi c(t) \)
  - \( (f \circ \varphi)(\xi, t) \) is now \( q \)-homogeneous w.r.t. \( \xi \)

Euler’s Homogeneous Function Theorem

\[
q(f \circ \varphi)(\xi, t) = \frac{\partial}{\partial \xi} \left[ (f \circ \varphi)(\xi, t) \right] \xi \quad \forall x \in V
\]

- \( V \subset \mathbb{R}^d \): domain where \( f \) maps \( x \) to \( \mathbb{R} \) (\( V = \mathbb{R}^d \) for a monomial)

\[
\frac{\partial}{\partial \xi} \left[ (f \circ \varphi)(\xi, t) \right] \xi^2
\]

\[
= \frac{\partial}{\partial \xi} \left[ (f \circ \varphi)(\xi, t) \right] \xi^2 + (f \circ \varphi)(\xi, t) 2\xi
\]

\[
= (2 + q) (f \circ \varphi)(\xi, t) \xi
\]

\( \leftarrow \) product rule

\( \leftarrow \) Euler’s homogeneous function theorem
Simplification for homogeneous functions

\[ \frac{\partial}{\partial \xi} \left[ (f \circ \varphi) (\xi, t) \xi^2 \right] = (2 + q) (f \circ \varphi) (\xi, t) \xi \quad (1) \]

- Integration over triangles:

\[
\int_{\mathscr{T}} f(x) \, dx = 2A_{\mathscr{T}} \int_0^1 \int_0^1 (f \circ \varphi) (\xi, t) \xi \, d\xi dt
\]

\[= \frac{2A_{\mathscr{T}}}{2 + q} \int_0^1 \int_0^1 \frac{\partial}{\partial \xi} \left[ (f \circ \varphi) (\xi, t) \xi^2 \right] d\xi dt \quad \leftarrow \text{using (1)} \]

\[= \frac{2A_{\mathscr{T}}}{2 + q} \int_0^1 (f \circ \varphi)(1, t) \, dt \quad \leftarrow \text{fundamental theorem of calculus} \]

\[= \frac{2A_{\mathscr{T}}}{2 + q} \int_0^1 (f \circ c)(t) \, dt \quad \leftarrow \varphi(1, t) = c(t) \]

- Equivalent to homogeneous numerical integration scheme, E. B. Chin et al., Comp Mech (2015)
Simplification for homogeneous functions

\[
\frac{\partial}{\partial \xi} \left[ (f \circ \varphi)(\xi, t) \xi^2 \right] = (2 + q) \left( f \circ \varphi \right)(\xi, t) \xi \tag{1}
\]

- Integration over curved triangles:

\[
\int_{\mathcal{T}} f(x) \, dx = \int_{0}^{1} \int_{0}^{1} (f \circ \varphi)(\xi, t) \xi \, d\xi \, c(t) \cdot c'(t) \, dt
\]

\[
= \frac{1}{2 + q} \int_{0}^{1} \int_{0}^{1} \frac{\partial}{\partial \xi} \left[ (f \circ \varphi)(\xi, t) \xi^2 \right] d\xi \, c(t) \cdot c'(t) \, dt
\]

\[
= \frac{1}{2 + q} \int_{0}^{1} (f \circ \varphi)(1, t) \, c(t) \cdot c'(t) \, dt
\]

\[
= \frac{1}{2 + q} \int_{0}^{1} (f \circ c)(t) \, c(t) \cdot c'(t) \, dt
\]

\[\leftarrow \text{using (1)}\]

\[\leftarrow \text{fundamental theorem of calculus}\]

\[\leftarrow \varphi(1,t) = c(t)\]

Integration of weakly singular functions

- Compute
  \[ \int_{\mathcal{T}} \frac{g(x)}{||x - x_c||^\beta} \, dx \]
  \(- 0 < \beta < 2 \) (integrand is weakly singular)

- Introduce SB parametrization with \( x_0 = x_c \) :
  \( \varphi(\xi, t) = x_c + \xi(c(t) - x_c) \)

\[
\int_{\mathcal{T}} \frac{g(x)}{||x - x_c||^\beta} \, dx = \int_{0}^{1} \left( \int_{0}^{1} \right) \left( g \circ \varphi \right)(\xi, t) \xi^{1-\beta} d\xi \frac{(c(t) - x_c) \cdot c'(t)}{||c(t) - x_c||^\beta} \, dt
\]

- If \( \beta = 1 \):

\[
\int_{\mathcal{T}} \frac{g(x)}{||x - x_c||} \, dx = \int_{0}^{1} \left( \int_{0}^{1} \right) \left( g \circ \varphi \right)(\xi, t) d\xi \frac{(c(t) - x_c) \cdot c'(t)}{||c(t) - x_c||} \, dt
\]

- If \( c(t) = [1 \quad t]^T \) and \( x_0 = 0 \), SB parametrization matches Duffy transformation (M. G. Duffy, SIAM J Numer Anal, 1982)
Integration of weakly singular functions

- Compute

\[ \int_{\mathcal{F}} \frac{g(x)}{\|x - x_c\|^{\beta}} \, dx \]

- 0 < \beta < 2 (integrand is weakly singular)

- If \( \beta \neq 1 \), convergence with SB parametrization is poor

- Introduce generalized SB parametrization: \( \varphi_\alpha(\xi, t) = x_c + \xi^\alpha (c(t) - x_c) \)

\[ \int_{\mathcal{F}} \frac{g(x)}{\|x - x_c\|^{\beta}} \, dx = \alpha \int_{0}^{1} \int_{0}^{1} (g \circ \varphi_\alpha)(\xi, t) \xi^{\alpha(2 - \beta) - 1} d\xi \frac{(c(t) - x_c) \cdot c'^\perp(t)}{\|c(t) - x_c\|^{\beta}} dt \]

- Polynomial integrand in \( \xi \) if \( \alpha \in \mathbb{Z}_\beta \), where \( \mathbb{Z}_\beta = \left\{ \alpha \in \mathbb{Z}_+ : \alpha(2 - \beta) \in \mathbb{Z}_+ \right\} \)

- Choose smallest \( \alpha \in \mathbb{Z}_\beta \)

- If \( c(t) = [1 \quad t]^T \) and \( x_0 = 0 \), the generalized SB parametrization matches generalized Duffy transformation

(S. E. Mousavi and NS, Comput Mech, 2010)

- Gauss-Jacobi rule also provides efficient \( \xi \)-integration
Integration of weakly singular functions

\[
\int_J \frac{g(x)}{\|x - x_c\|^\beta} \, dx = \alpha \int_0^1 \int_0^1 (g \circ \varphi_\alpha)(\xi, t) \, \xi^{\alpha(2-\beta)-1} \, d\xi \frac{(c(t) - x_c) \cdot c'(t)}{\|c(t) - x_c\|^\beta} \, dt
\]

- Generalized SB transformation eliminates \(\xi\)-singularities
- Near-singularities persist in the \(t\) direction
- Assume \(c(t)\) is affine: \(c(t) = \ell n + (t - t_0) t + x_c\)

\[
- \|c(t) - x_c\|^\beta = \left\{ \ell^2 + (\|t\|(t - t_0))^2 \right\}^{\beta/2} = (\ell^2 + \tau^2)^{\beta/2}
\]

\[
\cdot \tau = \|t\|(t - t_0) \quad dt = \frac{1}{\|t\|} \, d\tau
\]

\[
- (c(t) - x_c) \cdot c'(t) = \ell \|t\|
\]

\[
\int_J \frac{g(x)}{\|x - x_c\|^\beta} \, dx = \alpha \ell \int_{-\|t\| t_0}^{\|t\|(1-t_0)} \int_0^1 g(\varphi_\alpha) \xi^{\alpha(2-\beta)-1} \, d\xi \frac{1}{(\ell^2 + \tau^2)^{\beta/2}} \, d\tau
\]
Integration of weakly singular functions

\[ \int_{\mathcal{T}} \frac{g(x)}{\|x - x_c\|^{\beta}} \, dx = \alpha \ell \int_{-\|t\|t_0}^{\|t\|(1-t_0)} \int_0^1 g(\varphi_{\alpha}) \frac{\xi^{(2-\beta)-1}}{(\ell^2 + \tau^2)^{\beta/2}} \, d\xi \, \frac{1}{(\ell^2 + \tau^2)^{\beta/2}} \, d\tau \]

(H. Ma and N. Kamiya, Eng Anal Bound Elem, 2002)

- Define \( d\tilde{\tau} = \frac{\ell^{\beta-1}}{(\ell^2 + \tau^2)^{\beta/2}} \, d\tau \), solve for \( \tau \)
- Integrate in \( \tilde{\tau} \) to smooth near singularities (even if \( \beta \neq 1, 2, \text{ or } 3 \))
- If \( \beta = 1, 2 \text{ or } 3 \), near singularities are eliminated
- \( \alpha = 1, \beta = 1 \):

\[ \int_{\mathcal{T}} \frac{g(x)}{\|x - x_c\|^{\beta}} \, dx = \ell \int_{\ln(-\|t\|t_0 + \sqrt{\ell^2 + (-\|t\|t_0)^2})}^{\ln(\|t\|(1-t_0) + \sqrt{\ell^2 + [\|t\|(1-t_0)]^2})} \int_0^1 g(\varphi_{\alpha}) \, d\xi \, d\tilde{\tau} \]
Example: Integrating polynomials over polygons

- Convex and nonconvex polygons
- Two choices for $x_0$: 0 and vertex average
- Gauss points for $O(10^{-15})$ integration error

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\xi$-points</th>
<th>t-points</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>3</td>
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<tr>
<td>5</td>
<td>4</td>
<td>3</td>
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</tbody>
</table>

- Matches expected number of points
Example: Integrating non-polynomial functions over polygons

\[ f_{F1} = \frac{3}{4} e^{\frac{(9x-2)^2 + (9y-2)^2}{4}} + \frac{3}{4} e^{\frac{(9x-2)^2}{49}} \frac{9y-1}{10} + \frac{1}{2} e^{\frac{(9x-7)^2 + (9y-3)^2}{4}} + \frac{1}{5} e^{-(9x-4)^2 - (9y-7)^2} \]

\[ f_{F2} = \frac{1}{9} [\tanh(9y - 9x) + 1] \]

\[ f_{F3} = \frac{5}{4} + \cos \frac{27}{5} y \]

\[ \frac{6}{6[1 + (3x - 1)^2]} \]

(R. Franke, Tech Report, Naval Postgraduate School, 1979)
Example: Integrating non-polynomial functions over polygons

Comparisons with Gauss-Green (G-G) cubature (A. Sommariva and M. Vianello, BIT Numer Math, 2007)
Example: Time to generate integration rule over polygons

- Polygons inscribed in a circle
- Comparison to constrained Delaunay triangulation (CDT) and Gauss-Green (G-G)
- SBC is about 50% faster than G-G
- SBC is about 90% faster than CDT
Example: Integrating functions over curved regions

- $f_{C1}$: constant
- $f_{C2}$: 5th order polynomial
- $f_{C3}$: Franke function
- $f_{C4}$: exponential + trigonometric function

- $x_0$ at mean vertex coordinate
- $\xi$-direction: < 20 points/edge for machine precision error
- $t$-direction: < 40 points/edge for machine precision error
Example: Integrating singular functions ($\xi$ transformation)

Integrate:

\[
\begin{align*}
  f_{S1} &= 4 - 2x + y - x^2 + 2xy - 3y^2 + 3x^3 - 5x^2y + 5xy^2 - 4y^3 \\
  f_{S2} &= \exp\left(-\left[\left(\frac{x - 0.25}{0.4}\right)^2 + \left(\frac{y - 0.2}{0.7}\right)^2\right]^2\right)\cos^2(5x)\cos^2(5y)
\end{align*}
\]

Over:

Using:

- No transformation
- Generalized SBC rule
- Gauss-Jacobi rule
Example: Integrating singular functions ($\xi$ transformation)
Example: Integrating singular functions (t transformation)

Integrate:

\[ f_{S1} = \frac{4 - 2x + y - x^2 + 2xy - 3y^2 + 3x^3 - 5x^2y + 5xy^2 - 4y^3}{(x^2 + y^2)^{\frac{1}{4}}} \]

\[ f_{S2} = \frac{\exp\left(- \left[ \left( \frac{x - 0.25}{0.4} \right)^2 + \left( \frac{y - 0.2}{0.7} \right)^2 \right] \right) \cos^2(5x)\cos^2(5y)}{(x^2 + y^2)^{\frac{1}{4}}} \]

Over:

Using:

- No transformation
- \( \beta = 1 \) cancellation
- \( \beta = 2 \) cancellation
- \( \beta = 3 \) cancellation
Example: Integrating singular functions ($t$ transformation)
Application: Integrating Wachspress basis functions

- Integrate the product of derivatives of Wachspress basis functions on a maximal Poisson-disk sampled Voronoi mesh
- Methods of integration: SBC, Gauss-Green, triangulation
- SBC on-par with Gauss-Green and triangulation
Application: Extended finite elements for crack modeling

- Integrate stiffness matrix entries for elements with near-tip enrichment
  - $\Omega_1$ is crack-tip adjacent
  - $\Omega_2$ contains the crack-tip
  - $\Delta x = 0.001, 0.01, 0.1$

- Stiffness matrix entries contain a $r^{-\frac{1}{2}}$ singularity
Application: Extended finite elements for crack modeling

- $\beta = 1$ cancelling in the $t$-direction
- Generalized SB and Gauss-Jacobi provide similar accuracy per $\xi$-quadrature point
Application: Transfinite mean value interpolation (TMVI)

- Continuous counterpart to mean value coordinates. Given $g : \Gamma \rightarrow \mathcal{R}$ and $c(t)$, a parametric description of $\Gamma$, the TMVI is

$$u(x) = \frac{\int_0^1 g(c(t)) K(x, t) \, dt}{W(x)} \quad W(x) = \int_0^1 K(x, t) \, dt \quad K(x, t) = \frac{(c(t) - x) \cdot c'(t)}{\|c(t) - x\|^3}$$

- See C. Dyken and M. S. Floater, Comp Aided Geom Des (2009)

- $L_p$-distance fields are closely related to TMVI

$$\psi(x) = \left( \frac{1}{W_p(x)} \right)^{\frac{1}{p}} \quad W_p(x) = \int_0^1 \frac{(c(t) - x) \cdot c'(t)}{\|c(t) - x\|^{2+p}} \, dt$$

- $p = 1$ recovers TMVI weight function
- Distance function is recovered as $p \rightarrow \infty$

- See A. Belyaev et al., Comp Aided Des (2013)
Application: Transfinite mean value interpolation (TMVI)

- Find $L_2$ error in computing distance function versus $p$ in $L_p$-distance field
  - Granville egg domain
  - Compute $L_2$ error using SBC

![Graph showing relative $L_2$ error vs. p]

$p = 1$  
$p = 10$  
$p = 100$
Extension to three dimensions

- In 3D, SB transformation is $\varphi(\xi, u, v) = x_0 + \xi (c(u, v) - x_0)$
  - $c(u, v)$ is a surface describing the boundary of the region of integration

- On polyhedra, SB transformation can be applied recursively on the faces:

$$
\int f(x) \, dx = \sum_{i=1}^{\text{num faces}} \sum_{j=1}^{\text{num edges}_i} V_{ij} 6\int_0^1 \int_0^1 \int_0^1 (f \circ \varphi)(\xi, u, v) \xi^2 u \, d\xi \, du \, dv
$$

  - $V_{ij}$: volume of tetrahedron formed by vertices $x_0$, a point on face $i$, and the vertices of edge $j$

Szilassi polyhedron

$x_0 = 0$, 3x3x3 rule per tetrahedron

$x_0$ @ vertex, 3x3x3 rule per tetrahedron
Conclusions and outlook

- Introduced the scaled boundary cubature (SBC) scheme
  - Accuracy on par with other schemes such as Gauss-Green
  - Rule generation is fast, due to the simplicity of the SB transformation
  - Implementation is simple, only requires selection of scaling center \( x_0 \)
  - Positive cubature weights and points inside the domain with star convexity
  - Many valid domains of integration: nonconvex polygons, curved regions, non-simple regions, etc.
  - Efficient integration of weakly singular functions with \( \zeta \) and \( t \) transformations
  - SBC scheme is widely applicable: polygonal FEM, X-FEM, BEM, cutFEM, etc.

- Future research directions
  - Extension to three dimensions
  - Exploring applications in higher-order computational contact mechanics
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