An enhanced VEM formulation for plane elasticity

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The VEM dofs

A well-known peculiarity and advantage of VEM over other techniques is to allow for the use of arbitrary polygonal elements. For polygons with a high number of edges, this leads to a large number of dofs. For plane elasticity:

\[ k = 1 \]  
\[ k = 2 \]

\[ 7 \cdot 2 = 14 \text{ dofs} \]  
\[ 15 \cdot 2 = 30 \text{ dofs} \]
Displacements and strains parametrization

It is easy to see that the dofs used to characterize the displacement field are often redundant to characterize the assumed strain field.

<table>
<thead>
<tr>
<th>Edges</th>
<th>Disp. dofs</th>
<th>Def. dofs + RBM</th>
<th>Diff.</th>
<th>Disp. dofs</th>
<th>Def. dofs + RBM</th>
<th>Diff.</th>
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<td>12</td>
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</table>

As it is well-known, zero-energy deformation modes are penalized introducing stabilizing terms which approximate the energy of the virtual strains which do not project on the assumed (polynomial) strain modes.
Dofi-dofi

In the original VEM formulation the stabilization term, $K_s$, is usually calculated by means of the dofi-dofi formulation as

$$K_s = \tau \, \text{tr}(K_c) \left[ I - D \left( D^T D \right)^{-1} D^T \right],$$

(1)

which penalizes the components of the displacement fields which do not project (in a least square sense) on a polynomial base of order $k$. 
In order to ameliorate the accuracy of the calculated stress field, the authors recently proposed the use of the Recovery by Compatibility in Patches (RCP). RCP appears to be particularly well-suited for the VEM, as it necessitate to know the displacement field only at the patch boundary. In the case of numerous edges, RCP proved effective even with degenerate patches of a single element (RCP0).
The fact that $RCP0$ leads to an increase of accuracy and sometimes of the convergence rate, suggests that the strain field might be enhanced directly in the VEM formulation. Enhanced-VEM consists in:

- keep the classical VEM formulation of order $k$ as regards the displacement degrees of freedom at the boundary;
- use an adequate number of strain modes to project the virtual strains;
- eventually use an energy norm in the strain projection to guide the mode selection and avoid the generation of unnecessary internal dofs.

Additional internal dof $\mathbf{u} \rightarrow 7\cdot2 = 14$ dofs
$\mathbf{R}_{\text{BM}} - 3$ dofs
$\mathbf{\varepsilon} - 3$ dofs
$\mathbf{R}_{\text{BM}} - 9$ dofs
$\mathbf{u} \rightarrow 7 + 1)\cdot2 = 16$ dofs

$\mathbf{R}_{\text{BM}} - 3$ dofs
$\mathbf{\varepsilon} - 9$ dofs

Additional internal dof
Projection norm

In the standard VEM formulation the projector is defined as the unique function that minimizes

$$\left\| \varepsilon^P - \varepsilon(v_h) \right\|_{\text{norm}}$$

being $\varepsilon^P$ and $\varepsilon(v_h)$ the projected and the virtual strains. The minimization leads to

$$\int_{\Omega_E} [\varepsilon^P - \varepsilon(v_h)]^T \delta \varepsilon^P \, dA = 0 \quad \forall \delta \varepsilon^P \in P_p(\Omega_E).$$

(3)

If instead of (2) we adopt an energy norm we obtain

$$\int_{\Omega_E} [\varepsilon^P - \varepsilon(v_h)]^T C \delta \varepsilon^P \, dA = 0 \quad \forall \delta \varepsilon^P \in P_p(\Omega_E).$$

(4)

being $C$ the constitutive matrix.
Energy-based projection

It is at this point convenient to introduce a stress field representation such that

\[ \tilde{\mathbf{N}}^P \hat{\sigma} = \mathbf{C} \mathbf{N}^P \hat{\epsilon}, \]  

(5)

Then, the minimization leads to

\[ \hat{\sigma} = \mathcal{G}^{-1} \mathcal{B}, \]  

(6)

being

\[ \mathcal{G} = \int_{\Omega_E} (\tilde{\mathbf{N}}^P)^T \mathbf{C}^{-1} \tilde{\mathbf{N}}^P dA, \quad \mathcal{B}[\tilde{\mathbf{V}}^T, \hat{\mathbf{V}}^T]^T = \tilde{\mathcal{B}}\tilde{\mathbf{V}} + \hat{\mathcal{B}}\hat{\mathbf{V}}, \]  

(7)

where, denoting as \( \mathbf{L} \) the compatibility operator

\[ \tilde{\mathcal{B}}\tilde{\mathbf{V}} = \int_{\partial\Omega_E} (\mathbf{N}_E^T \tilde{\mathbf{N}}^P)^T \mathbf{N}^V d\tilde{\mathbf{V}}, \quad \hat{\mathcal{B}}\hat{\mathbf{V}} = -\int_{\Omega_E} (\mathbf{L}^T \tilde{\mathbf{N}}^P)^T \mathbf{v}_h dA. \]  

(8)
Consistent stiffness matrix

Then, as in the standard VEM, the consistent part of the stiffness matrix is calculated as

\[ K_c = B^T G^{-T} \left( \int_{\Omega_E} (\mathbf{N}^P)^T C^{-1} \mathbf{N}^P dA \right) G^{-1} B. \]  

(9)

The following points shall be noticed:

- in such formulation \( \tilde{\mathbf{N}}^P \) collects stress modes as in mixed-stress formulations;

- according to the E-VEM philosophy, for polygons with numerous edges, the stress field can be of higher order than \( k - 1 \);

- it is possible to preselect only equilibrated stress-modes instead of the full polynomial expansion for cases with null loading, so avoiding internal dofs.
Enhanced VEM

Stress modes

In fact, the internal dofs are introduced in order to compute

$$\hat{B} \hat{V} = - \int_{\Omega} (L^T \tilde{N}^P)^T \mathbf{v}_h dA. \quad (10)$$

but, as $\tilde{N}^P$ now collects stress modes, for null loading the result is known to be null \textit{a priori}. Stress modes can be extracted by the Airy's stress function, $\phi$

$$\tilde{N}^P_i = \begin{bmatrix} \tilde{N}^P_{i,1} \\ \tilde{N}^P_{i,2} \\ \tilde{N}^P_{i,3} \end{bmatrix}, \quad \tilde{N}^P_{i,1} = \frac{\partial^2 \phi}{\partial y^2}, \quad \tilde{N}^P_{i,2} = \frac{\partial^2 \phi}{\partial x^2}, \quad \tilde{N}^P_{i,3} = - \frac{\partial^2 \phi}{\partial x \partial y} \quad (11)$$

In the following, results obtained by using a complete polynomial expansion are marked as $UCP$, while those obtained considering only div-free stress modes are denoted as $DFP$. Also $HYP$ is a hybrid approach which is $UCP$ for low-order stress modes and $DFP$ for high order ones.
Stabilization term

In the contest of E-VEM, due to the presence of an higher number of stress/strain modes, it is often possible to avoid stabilization. Numerical experimentation led to the following results:

<table>
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<tr>
<th>$m$ (n. vertexes)</th>
<th>$k$</th>
<th>$p$</th>
<th>Enhancement</th>
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<td>27</td>
<td>$s = 3$</td>
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Numerical tests have been run on the following meshes:

- **(a) QUAD**
- **(b) RHOM**
- **(c) HEXA**
- **(d) WEBM**
- **(e) DODE**

In particular, mesh **DODE** has been built in order to be representative of polygons with an high number of edges.
Numerical tests

• Load case A — null distributed volume forces

\[ \mathbf{v}(x, y) = \begin{bmatrix} -\frac{x^6}{80} + \frac{x^4 y^2}{2} - \frac{13}{16} x^2 y^4 + \frac{3}{40} y^6 \\ \frac{xy^5}{4} - \frac{5}{12} x^3 y^3 \end{bmatrix}, \quad (12) \]

leading to \( \mathbf{b}(x, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

• Load case B — non-null distributed volume forces

\[ \mathbf{v}(x, y) = \begin{bmatrix} x \sin(\pi x) \sin(\pi y) \\ y \sin(\pi x) \sin(\pi y) \end{bmatrix}, \quad (13) \]

leading to \( \mathbf{b}(x, y) = \begin{bmatrix} \frac{11}{3} \pi^2 x \sin(\pi x) \sin(\pi y) - \frac{5}{3} \pi^2 y \cos(\pi x) \cos(\pi y) - 7\pi \cos(\pi x) \sin(\pi y) \\ \frac{11}{3} \pi^2 y \sin(\pi x) \sin(\pi y) - \frac{5}{3} \pi^2 x \cos(\pi x) \cos(\pi y) - 7\pi \cos(\pi y) \sin(\pi x) \end{bmatrix} \).
Figure 1: Convergence for Load case A: (a) QUAD, (b) RHOM.
Figure 2: Convergence for Load case A: (a) HEXA, (b) WEBM, and (c) DODE.
Numerical tests

Test B I

Figure 3: Convergence for Load case B: (a) QUAD, (b) RHOM.
Figure 4: Convergence for Load case B: (a) HEXA, (b) WEBM, and (c) DODE.
Numerical tests

Example of stress field for DODE

(a) VEM

(b) UCP $p = 2$

(c) UCP $p = 3$

(d) Exact
Conclusions

An enhanced version of VEM has been proposed:

- the enhancement is based on an enrichment of the internal stress representation, while keeping a low-order displacement interpolation along edges;
- the technique proved effective in increasing accuracy for polygons with many edges;
- in many cases, by using such enhancement, it is possible to avoid the stabilization;
- by introducing an energy norm in the calculation of the VEM projector, it is possible to avoid the introduction of internal dofs for null distributed loading.